

Perfect squares have at most five divisors close to its square root

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Abstract

In this paper, we consider a conjecture of Erdős and Rosenfeld when the number is a perfect square. In particular, we show that every perfect square n can have at most five divisors between $\sqrt{n} - c\sqrt[4]{n}$ and $\sqrt{n} + c\sqrt[4]{n}$.

1 Introduction and main result

In [4], Erdős and Rosenfeld considered the differences between the divisors of a positive integer n . They exhibited infinitely many integers with four “small” differences and posed the question that any positive integer can have at most a bounded number of “small” differences. Specifically, they asked

Question 1 *Is there an absolute constant K , so that for every c , the number of divisors of n between \sqrt{n} and $\sqrt{n} + c\sqrt[4]{n}$ is at most K for $n > n_0(c)$?*

In this paper, we answer the above question when n is a perfect square. In particular, we have

Theorem 1 *For every $c \geq 3$, any perfect square n can have at most five divisors between $\sqrt{n} - c\sqrt[4]{n}$ and $\sqrt{n} + c\sqrt[4]{n}$ for $n > e^{Cc^6(\log c)^5}$ where C is some sufficiently large constant independent of c .*

This answers question 1 for perfect squares with $K = 3$. Based on the proof of Theorem 1, every example where a perfect square n has three divisors between \sqrt{n} and $\sqrt{n} + c\sqrt[4]{n}$ comes from solutions to Pell equations. For example, consider the Pell equation $X^2 - 2Y^2 = 2$. It has solutions (X_k, Y_k) generated by $X_k + \sqrt{2}Y_k = (3 + 2\sqrt{2})^k(2 + \sqrt{2})$. Then one can verify that X_k are even, Y_k are odd and $(X_k - 2)(X_k + 2) = 2(Y_k - 1)(Y_k + 1)$. Now consider the integers $n = (X_k - 2)^2(X_k + 2)^2 = 4(Y_k - 1)^2(Y_k + 1)^2$. It has divisors $(X_k - 2)(X_k + 2)$, $(X_k + 2)^2$, $2(Y_k + 1)^2$ that are between \sqrt{n} and $\sqrt{n} + 5\sqrt[4]{n}$. This shows that $K = 3$ is the best possible constant for question 1 to be true with perfect squares.

2 Initial transformation

Suppose $N^2 = (N - d_1)(N + e_1) = (N - d_2)(N + e_2) = \dots = (N - d_r)(N + e_r)$ where $N, N - d_i, N + e_j$ are all the divisors of N^2 that lie in $[N - cN^{1/2}, N + cN^{1/2}]$ where $1 \leq d_1 < d_2 < \dots < d_r \leq cN^{1/2}$ and $1 \leq e_1 < e_2 < \dots < e_r \leq cN^{1/2}$ are positive integers. Observe that $N^2 = (N - d_i)(N + e_i)$ which gives $e_i d_i = (e_i - d_i)N$. So we must have $e_i > d_i$ and say $e_i = d_i + l_i$ for some positive integer l_i . Hence $(d_i + l_i)d_i = l_i N$ which gives $d_i^2 + l_i d_i = l_i N$. Multiply both sides by four and add $l_i^2 + 4N^2$ to both sides, we have $(2d_i + l_i)^2 + (2N)^2 = (2N + l_i)^2$. Also from $(d_i + l_i)d_i = l_i N$, we have

$$l_i = \frac{d_i^2}{N - d_i} \leq \frac{c^2 N}{N - cN^{1/2}} \leq 2c^2 \quad (1)$$

for $N \geq 4c^2$.

3 Pythagorean triples

Thus we have a Pythagorean triple $2d_i + l_i, 2N, 2N + l_i$. It is well-known that all the solutions to the Pythagorean equation are parametrized by $\lambda(u^2 - v^2), \lambda(2uv), \lambda(u^2 + v^2)$ for some positive integers λ and $u > v$.

Case 1: $2d_i + l_i = \lambda_i(u_i^2 - v_i^2)$, $2N = \lambda_i(2u_i v_i)$ and $2N + l_i = \lambda_i(u_i^2 + v_i^2)$. Subtracting the last two equations, we have $l_i = \lambda_i(u_i - v_i)^2 \leq 2c^2$ by (1). Hence $u_i - v_i \leq \sqrt{2}c$, $\lambda_i \leq 2c^2$ and $2N$ can be written as $2\lambda_i u_i v_i$. By adding or subtracting the three equations, we also have $2(N - d_i) = 2\lambda_i v_i^2$ and $2(N + e_i) = 2(N + d_i + l_i) = 2\lambda_i u_i^2$.

Case 2: $2N = \lambda_i(u_i^2 - v_i^2)$, $2d_i + l_i = \lambda_i(2u_i v_i)$ and $2N + l_i = \lambda_i(u_i^2 + v_i^2)$. Subtracting the first and the last equations, we have $l_i = 2\lambda_i v_i^2 \leq 2c^2$ by (1). Hence $v_i \leq c$, $\lambda_i \leq c^2$ and $2N$ can be written as $\lambda_i(u_i - v_i)(u_i + v_i)$. By adding or subtracting the three equations, we also have $2(N - d_i) = \lambda_i(u_i - v_i)^2$ and $2(N + e_i) = 2(N + d_i + l_i) = \lambda_i(u_i + v_i)^2$.

In either case, $2N = \mu_i x_i y_i$, $2(N - d_i) = \mu_i x_i^2$ and $2(N + e_i) = \mu_i y_i^2$ with $1 \leq y_i - x_i \leq 2c$, $\mu_i = \lambda_i$ or $2\lambda_i$ and $\mu_i \leq 4c^2$.

4 Almost squares

Now we claim that the μ_i are distinct if $N > 32c^6$. Suppose not, say $\mu_i = \mu_j$ for some $1 \leq i < j \leq r$. Then $\mu_i x_i y_i = 2N = \mu_j x_j y_j$ implies $x_i y_i = x_j y_j = \frac{2N}{\mu_i}$. Numbers like $\frac{2N}{\mu_i}$ that can be factored as $x_i y_i$ and $x_j y_j$ with x_i close to y_i and x_j close to y_j are called almost squares of type 2 and have been studied by the author in [1], [2] and [3] for example. If $x_i = x_j$, then $2(N - d_i) = \mu_i x_i^2 = \mu_j x_j^2 = 2(N - d_j)$ which contradicts $d_i < d_j$. Without loss of generality, assume $x_i < x_j$. Then we must have $x_i < x_j < y_j < y_i$. Let $y_j = n$, $x_j = n - f$, $x_i = n - g$, $y_i = n + h$ for some positive integers f, g, h . Since $1 \leq y_i - x_i, y_j - x_j \leq 2c$, $f, g, h \leq 2c$. We have $n(n - f) = (n - g)(n + h)$ which implies $(f + h - g)n = gh$. Since $gh > 0$, we must have $f + h - g > 0$. Therefore

$$\sqrt{\frac{2N}{4c^2}} \leq \sqrt{\frac{2N}{\mu_i}} \leq n \leq (f + h - g)n = gh \leq (2c)^2$$

which contradicts $N > 32c^6$.

5 Simultaneous Pell equations

Summing up, if $N^2 = (N - d_1)(N + e_1) = (N - d_2)(N + e_2) = \dots = (N - d_r)(N + e_r)$ with $1 \leq d_1 < d_2 < \dots < d_r \leq cN^{1/2}$ and $1 \leq e_1 < e_2 < \dots < e_r \leq cN^{1/2}$ and $N > 32c^6$, then we have $2N = \mu_1 x_1 y_1 = \mu_2 x_2 y_2 = \dots = \mu_r x_r y_r$ where μ_i 's are distinct, $\mu_i \leq 4c^2$ and $1 \leq y_i - x_i \leq 2c$. Let $y_i := x_i + c_i$ for some integer $1 \leq c_i \leq 2c$. Then

$$8N = \mu_1(2x_1)(2x_1 + 2c_1) = \mu_2(2x_2)(2x_2 + 2c_2) = \dots = \mu_r(2x_r)(2x_r + 2c_r).$$

Suppose $r \geq 3$ for otherwise Theorem 1 is true. Now $\mu_1(2x_1 + c_1)^2 - \mu_1 c_1^2 = \mu_1(2x_1)(2x_1 + 2c_1) = \mu_2(2x_2)(2x_2 + 2c_2) = \mu_2(2x_2 + c_2)^2 - \mu_2 c_2^2$. This leads to the Pell equation

$$\mu_1(2x_1 + c_1)^2 - \mu_2(2x_2 + c_2)^2 = \mu_1 c_1^2 - \mu_2 c_2^2. \quad (2)$$

Similarly,

$$\mu_1(2x_1 + c_1)^2 - \mu_3(2x_3 + c_3)^2 = \mu_1c_1^2 - \mu_3c_3^2. \quad (3)$$

Lemma 1 For $i \neq j$, $\mu_i c_i^2 \neq \mu_j c_j^2$.

Proof: Suppose $\mu_i c_i^2 = \mu_j c_j^2$ for some $1 \leq i < j \leq r$. Then

$$\frac{2x_i}{c_i} \left(\frac{2x_i}{c_i} + 1 \right) = \frac{\mu_i(2x_i)(2x_i + c_i)}{\mu_i c_i^2} = \frac{\mu_j(2x_j)(2x_j + c_j)}{\mu_j c_j^2} = \frac{2x_j}{c_j} \left(\frac{2x_j}{c_j} + 1 \right)$$

which implies $\frac{x_i}{c_i} = \frac{x_j}{c_j} = \Lambda > 0$ say. Then

$$\mu_i x_i^2 \left(1 + \frac{1}{\Lambda} \right) = \mu_i x_i (x_i + c_i) = \mu_j x_j (x_j + c_j) = \mu_j x_j^2 \left(1 + \frac{1}{\Lambda} \right).$$

Hence $\mu_i x_i^2 = \mu_j x_j^2$. But recall $2(N - d_i) = \mu_i x_i^2$ and $2(N - d_j) = \mu_j x_j^2$. This implies $2(N - d_i) = 2(N - d_j)$ which contradicts $d_i < d_j$.

By Lemma 1, we have $\mu_1(\mu_1 c_1^2 - \mu_3 c_3^2) \neq \mu_1(\mu_1 c_1^2 - \mu_2 c_2^2)$. By a result of Turk [5, Proposition 3], the solutions to both (2) and (3) satisfy

$$2x_1 + c_1 < e^{C(4c^2)^2(\log 4c^2)^3(4c^2 \log 4c^2) \log(4c^2 \log 4c^2)} \leq e^{C' c^6 (\log c)^5}$$

for some large constants C and C' . But $x_1 + c_1 = y_1 > \sqrt{\frac{2N}{\mu_1}} \geq \sqrt{\frac{2N}{4c^2}}$. This gives a contradiction if $N > e^{C'' c^6 (\log c)^5}$ with C'' sufficiently large. Therefore if $N^2 = (N - d_1)(N + e_1) = (N - d_2)(N + e_2) = \dots = (N - d_r)(N + e_r)$ with $1 \leq d_1 < d_2 < \dots < d_r \leq cN^{1/2}$ and $1 \leq e_1 < e_2 < \dots < e_r \leq cN^{1/2}$ and $N > e^{C'' c^6 (\log c)^5}$, then $r \leq 2$.

6 A catch

The above argument is almost correct except that when applying Turk's result to simultaneous Pell equations, one requires the coefficients μ_1, μ_2, μ_3 in (2) and (3) to be squarefree. So we need to modify our argument. Suppose $\mu_1 = \tilde{\mu}_1 t_1^2, \mu_2 = \tilde{\mu}_2 t_2^2, \mu_3 = \tilde{\mu}_3 t_3^2$ where t_i^2 is the largest perfect square that divides μ_i and hence $\tilde{\mu}_i$ is squarefree. Then since $\mu_i \leq 4c^2$, we have $\tilde{\mu}_i \leq 4c^2$ and $1 \leq t_i \leq 2c$ for $i = 1, 2, 3$. The Pell equations (2) and (3) become

$$\tilde{\mu}_1 [t_1(2x_1 + c_1)]^2 - \tilde{\mu}_2 [t_2(2x_2 + c_2)]^2 = \tilde{\mu}_1 t_1^2 c_1^2 - \tilde{\mu}_2 t_2^2 c_2^2, \quad (4)$$

and

$$\tilde{\mu}_1 [t_1(2x_1 + c_1)]^2 - \tilde{\mu}_3 [t_3(2x_3 + c_3)]^2 = \tilde{\mu}_1 t_1^2 c_1^2 - \tilde{\mu}_3 t_3^2 c_3^2. \quad (5)$$

Turk's result requires $\tilde{\mu}_1 \neq \tilde{\mu}_2$ and $\tilde{\mu}_1 \neq \tilde{\mu}_3$. Suppose on the contrary $\tilde{\mu}_1 = \tilde{\mu}_2$ (the case $\tilde{\mu}_1 = \tilde{\mu}_3$ is similar). Tracing back, as $2N = \mu_1 x_1 y_1 = \mu_2 x_2 y_2$, we have

$$2N = \tilde{\mu}_1 t_1^2 x_1 y_1 = \tilde{\mu}_2 t_2^2 x_2 y_2 \text{ which implies } (t_1 x_1)(t_1 y_1) = (t_2 x_2)(t_2 y_2).$$

Note that $t_1 x_1 < t_1 y_1$ and $t_2 x_2 < t_2 y_2$. There are two possibilities.

Case 1: $t_1 x_1 = t_2 x_2$. Then we also have $t_1 y_1 = t_2 y_2$. Recall $y_i = x_i + c_i$. So $t_1(x_1 + c_1) = t_2(x_2 + c_2)$ which implies $t_1 c_1 = t_2 c_2$ and hence $\tilde{\mu}_1 t_1^2 c_1^2 = \tilde{\mu}_2 t_2^2 c_2^2$ or $\mu_1 c_1^2 = \mu_2 c_2^2$ which contradicts Lemma 1.

Case 2: $t_1x_1 \neq t_2x_2$. We are in the almost square situation in section 4. By the same argument except replacing $2c$ by $4c^2$ (since $t_iy_i - t_ix_i = t_i(y_i - x_i) \leq 2c \cdot 2c$), we have a contradiction if $N > 512c^{10}$.

Therefore, since in Theorem 1 we have $n = N^2 > e^{Cc^6(\log c)^5}$, we do have $\tilde{\mu}_1 \neq \tilde{\mu}_2$ and $\tilde{\mu}_1 \neq \tilde{\mu}_3$. By Lemma 1, one can also check that $\tilde{\mu}_1(\tilde{\mu}_1t_1^2c_1^2 - \tilde{\mu}_3t_3^2c_3^2) \neq \tilde{\mu}_1(\tilde{\mu}_1t_1^2c_1^2 - \tilde{\mu}_2t_2^2c_2^2)$. Consequently, we can apply Turk's result to the Pell equations (4) and (5) and get the same result as in section 5 which finishes the proof of Theorem 1.

References

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